COMPLEX NUMBERS

Reference :

James G., Modern Engineering Mathematics, Chapter 2.

1 INTRODUCTION

1.1 How complex numbers arise

The equation of motion for a mass *m* hanging on a spring with 'spring constant' *k* is,

$$m\frac{d^2x}{dt^2} + k\,x \;=\; 0$$

where x is the displacement from the equilibrium position. If we substitute an assumed solution of the form $x = x_0\, e^{\lambda t}$, we obtain $(m\lambda^2 + k)\,x_0\, e^{\lambda t} = 0$, which gives $\lambda = \sqrt{-k/m}$. As *m* and *k* are both positive quantities and the square root of a negative number does not exist in the real number system, we need to introduce another type of number if the approach is to remain valid. That type of number is called an *imaginary* number. The simplest imaginary number is $\sqrt{-1}$ and is given the symbol *i* or *j* (we shall normally use *i*).

1.2 A bit of history

A 16th century Italian called Girolamo Cardano was the first to use imaginary numbers to solve a real equation. He was the illegitimate son of Fabio Cardano and Chiara Micheri, a widow of uncertain age who was both ignorant and irascible (hardly an ideal mother for a budding mathematician). Having been cured of the impotence which had afflicted him throughout his youth (one wonders how), Cardano married at the age of thirty and had two sons and a daughter. Fortified by this start in life, he showed that the solution of the cubic equation $x^3 + bx + c = 0$ is,

$$x \;=\; \sqrt[3]{-\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}} \;+\; \sqrt[3]{-\frac{c}{2} - \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}}$$

Cardano and others tried to use this to solve $x^3 - 15x - 4 = 0$ but found,

$$x = \sqrt[3]{2 + \sqrt{-121}} \;+\; \sqrt[3]{2 - \sqrt{-121}}$$

Just suppose, they thought, that we can treat $\sqrt{-1}$ like any other number. Then:

$$x \;=\; \sqrt[3]{2 + 11\sqrt{-1}} \;+\; \sqrt[3]{2 - 11\sqrt{-1}}$$

By simple algebra, $2 + 11\sqrt{-1} = \left(2 + \sqrt{-1}\right)^3$, and $2 - 11\sqrt{-1} = \left(2 - \sqrt{-1}\right)^3$, so they arrived at x = 4 which is indeed a solution of $x^3 - 15x - 4 = 0$. Hence, $\sqrt{-1}$ does seem to be a valid concept. But what does it mean?

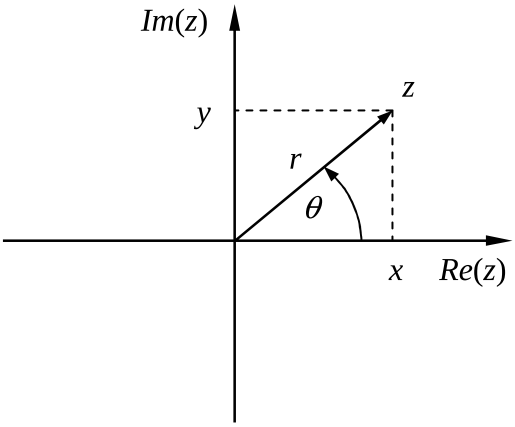
1.3 Definition of a complex number

We define the complex number z to be the quantity,

$$z = x + iy$$

where x and y are real numbers and $i = \sqrt{-1}$. x is called the *real part* of z and is written $Re(z)$ and y is called the *imaginary part* of z and is written $Im(z)$. Note that $Im(z)$ does not include the i. Two complex numbers are equal if both their real and imaginary parts are equal.

We can plot a complex number as a point (x, y) in the *complex plane* (known as the *Argand diagram*) with x measured on the *real axis* and y measured on the *imaginary axis*.



Alternatively, we can use polar coordinates (r, θ). In this case,

$$z = r\cos\theta + i\,r\sin\theta$$

where $r = |z|$ is the *magnitude* (or *modulus*) of z and $\theta = \arg z$ is the *argument* of z. From the Argand diagram we have,

$$r = |z| = \sqrt{x^2 + y^2} \qquad \left(\text{NOT } \sqrt{x^2 + (iy)^2} = \sqrt{x^2 - y^2}\right)$$

$$\theta = \arg z = \tan^{-1}\left(\frac{y}{x}\right)$$

Because (r, θ) and $(r, \theta + 2\pi)$ represent the same point, a convention is used to determine arg z uniquely by restricting its range so that $0 \le \theta < 2\pi$.

Complex numbers are called numbers because when we do mathematics with them using the rule that $i = \sqrt{-1}$, we find that they obey all the rules of arithmetic and algebra. Also, many of the formulae in the Mathematics Data Book (such as the trigonometric identities and the power series expansions) are as valid for complex variables as for real ones.

1.4 The theorems of Euler and de Moivre

Consider the expression $e^{i\theta}$. If complex numbers really do obey the rules of algebra, then we can write $e^{i\theta}$ as a Maclaurin series,

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \ldots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)$$

Hence, we get Euler's theorem :

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Multiplying by *r* we obtain the polar representation of *z* in its most useful form :

$$\boxed{z = r\cos\theta + ir\sin\theta = re^{i\theta}}$$

Application of Euler's theorem also gives the rather startling result of de Moivre's theorem,

$$\boxed{(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)}$$

2 COMPLEX NUMBER ARITHMETIC

2.1 The basic operations

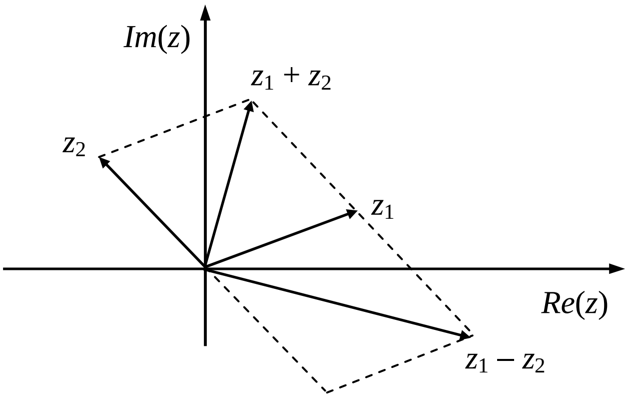
Addition (or subtraction) of complex numbers is achieved by independently adding (or subtracting) the real and complex parts. Hence, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$:

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$$

If the complex numbers are in polar form, addition and subtraction is best carried out by first converting to cartesians. Hence, if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$:

$$z_1 \pm z_2 = (r_1\cos\theta_1 \pm r_2\cos\theta_2) + i(r_1\sin\theta_1 \pm r_2\sin\theta_2)$$

On the Argand diagram, complex numbers add and subtract like 2D vectors :



Multiplication and division are best carried out in polar form :

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \quad \rightarrow \quad |z_1 z_2| = r_1 r_2 \,, \quad \arg(z_1 z_2) = (\theta_1+\theta_2)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \quad \rightarrow \quad \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} \,, \quad \arg\left(\frac{z_1}{z_2}\right) = (\theta_1-\theta_2)$$

In cartesian form the operations are messier :

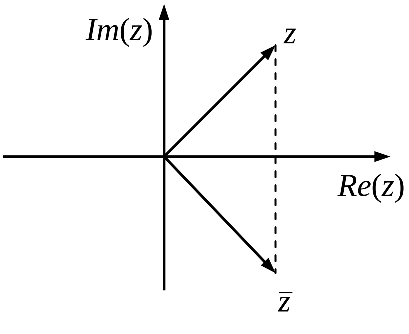
$$z_1 z_2 = (x_1+iy_1)(x_2+iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

$$\frac{z_1}{z_2} = \frac{(x_1+iy_1)}{(x_2+iy_2)} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)} = \frac{(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)}{x_2^2+y_2^2}$$

Note that for division, the real and imaginary parts are obtained by multiplying top and bottom by $(x_2 - iy_2)$ as this turns the denominator into a real number.

2.2 The complex conjugate

Complex arithmetic is often simplified by working in terms of the conjugate of a complex number. The complex conjugate of $z = x + iy$ is written $\bar{z}$ or z^* and is defined by $\bar{z} = x - iy = re^{-i\theta}$. On the Argand diagram :



From the definition it follows that,

$$z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 = |z|^2$$

$$z\bar{z} = (re^{i\theta})(re^{-i\theta}) = r^2 = |z|^2$$

The complex conjugate allows us to find the reciprocal of a complex number and hence do division (as explained above),

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$$

For example,

$$\frac{1}{(3+4i)} = \frac{(3-4i)}{(3+4i)(3-4i)} = \frac{(3-4i)}{9+16} = \frac{3}{25} - i\frac{4}{25}$$

To find the congugate of the sum of two complex numbers,

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1+x_2) + i(y_1 + y_2)$$

we take the conjugate of each side to give,

$$\overline{z_1 + z_2} = (x_1+x_2) - i(y_1 + y_2) = \bar{z}_1 + \bar{z}_2$$

To find the conjugate of the product of two complex numbers,

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

we take the conjugate of each side to give,

$$\overline{z_1 z_2} = r_1 r_2 e^{-i(\theta_1+\theta_2)} = \bar{z}_1 \bar{z}_2$$

2.3 Powers and roots

Raising a complex number to a power is easily carried out in polar form. Thus,

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} \quad \rightarrow \quad |z^n| = r^n, \quad \arg(z^n) = n\theta$$

Finding the square root, cube root and higher roots of complex numbers is more difficult and requires attention to detail. We start by noting that if n is any integer,

$$e^{in2\pi} = \cos(n2\pi) + i\sin(n2\pi) = 1$$

Hence, we can add integral multiples of 2π to the argument of z without changing its value.

Suppose we wish to find $z = (8i)^{1/3}$; *i.e.*, solve the cubic equation $z^3 - 8i = 0$. Now cubic equations have three roots and if we only get one something has gone wrong. We therefore proceed by writing 8*i* in polar form but with $n2\pi$ added to the argument,

$$8\,i \;=\; 8e^{i(\pi/2 \,+\, n2\pi)}$$

The addition of the $n2\pi$ does not change the value but when we find the cube root (which will involve dividing the argument by 3) none of the roots will go missing. Thus,

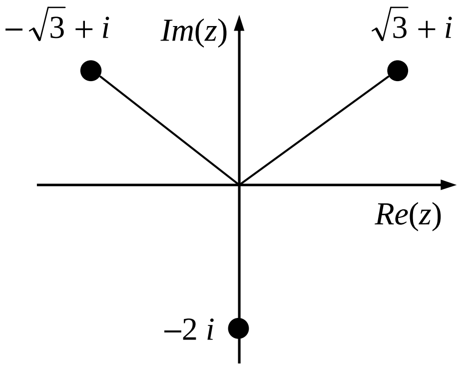
$$z \;=\; (8i)^{1/3} \;=\; 2e^{i(\pi/6 \,+\, n2\pi/3)}$$

The three roots are obtained by setting *n* equal to 0, 1 and 2. Other values of *n* simply give repetition. Hence,

$$z \;=\; (8i)^{1/3} \;=\; 2e^{i\pi/6} \quad \text{or} \quad 2e^{i5\pi/6} \quad \text{or} \quad 2e^{i3\pi/2}$$

$$=\; (\sqrt{3} + i) \quad \text{or} \quad (-\sqrt{3} + i) \quad \text{or} \quad (-2i)$$

On the Argand diagram, the three roots are symmetrical,

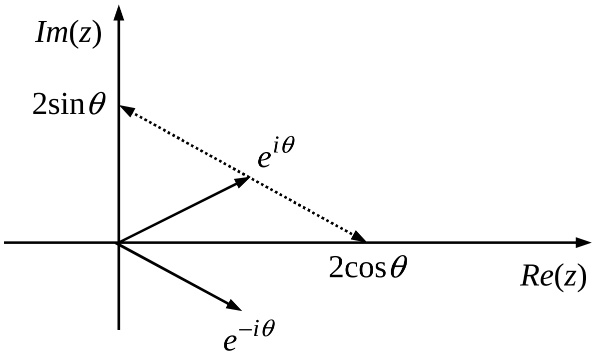


2.4 cos θ and sin θ

From Euler’s theorem :

$$\cos\theta \;=\; \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta \;=\; \frac{e^{i\theta} - e^{-i\theta}}{2i}$$



Manipulations of trigonometric functions are usually much easier using $e^{i\theta}$. For example,

$$e^{i(a+b)} = e^{ia} e^{ib}$$

Hence, from Euler's theorem,

$$\cos(a+b) + i\sin(a+b) = (\cos a + i\sin a)(\cos b + i\sin b)$$

Multiplying out and equating real and imaginary parts we obtain the trigonometric identities,

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

2.5 cosh θ and sinh θ

The relationships between the trigonometric and hyperbolic functions are established as follows :

$$\cos(i\theta) = \frac{e^{i(i\theta)} + e^{-i(i\theta)}}{2} = \frac{e^{-\theta} + e^{\theta}}{2} = \cosh\theta$$

$$\sin(i\theta) = \frac{e^{i(i\theta)} - e^{-i(i\theta)}}{2i} = \frac{e^{-\theta} - e^{\theta}}{2i} = \frac{i(e^{\theta} - e^{-\theta})}{2} = i\sinh\theta$$

2.6 Complex numbers are 2D numbers

Engineers expect to use numbers to quantify things so what can you quantify with a complex number? It helps to think back to the concept of negative numbers (developed earlier by the same group of Italians, also to solve polynomial equations). No-one knew what a negative number was, but Galileo observed that it becomes physically relevant if it is attributed to a quantity such as velocity which has direction. Multiplying a body's velocity by -1 means rotating its direction by 180°. Positive and negative numbers became accepted as numbers laid out on a one-dimensional 'number line'. Argand sensed that $\sqrt{-1}$ is somewhere between -1 and $+1$, so proposed that complex numbers are 2D numbers, where multiplication by $\sqrt{-1}$ means rotation by 90° as laid out in the Argand diagram. Since then 4D and even 8D numbers have been shown to obey the rules of algebra but are of little use to engineers. Really, it is negative numbers that should challenge our intuition because we are required to accept that numbers can have a dimension: after that, the idea of multi-dimensional numbers is just an extension.

3 EXAMPLES AND APPLICATIONS

3.1 Using the complex conjugate relations

(i) Find $\left|\frac{1}{1+e^{i\theta}}\right|$.

As we only need the magnitude of the complex number, we can use the result previously derived that $z\bar{z} = |z|^2$. Remembering also that $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$, we obtain,

$$\left|\frac{1}{1+e^{i\theta}}\right|^2 = \frac{1}{(1+e^{i\theta})(1+e^{-i\theta})} = \frac{1}{1+1+e^{i\theta}+e^{-i\theta}} = \frac{1}{2+2\cos\theta}$$

$$\left|\frac{1}{1+e^{i\theta}}\right| = (2+2\cos\theta)^{-1/2}$$

(ii) Find the real and imaginary parts of $\frac{1}{1+e^{i\theta}}$.

Multiply top and bottom by the complex conjugate of the denominator to give,

$$\frac{1}{1+e^{i\theta}} = \frac{(1+e^{-i\theta})}{(1+e^{i\theta})(1+e^{-i\theta})} = \frac{1+\cos\theta - i\sin\theta}{2+2\cos\theta} = \frac{1}{2} - i\left(\frac{\sin\theta}{2+2\cos\theta}\right)$$

3.2 Finding the inverse function of a complex number

Find $z = \sin^{-1}(i)$.

The trick is to invert the equation. Thus, taking the sine of both sides,

$$i = \sin(z) = \sin(x+iy) = \sin x\cos(iy) + \cos x\sin(iy)$$

$$= \sin x\cosh y + i\cos x\sinh y$$

Equating real and imaginary parts :

$$\sin x\cosh y = 0$$
$$\cos x\sinh y = 1$$

From the first equation (noting $\cosh y > 0$) :

$$x = n\pi \quad \text{where} \quad n = 0, \pm 1, \pm 2, ...$$

From the second equation :

$$\text{when } x = 0, \pm 2\pi, \pm 4\pi, \ldots \quad y = \sinh^{-1}(1) \;=\; 0.881$$
$$\text{when } x = \pm\pi, \pm 3\pi, \ldots \quad y = \sinh^{-1}(-1) \;=\; -0.881$$

Hence the solutions are :

$$z \;=\; n\pi \;+\; 0.881\,i \qquad (n \;=\; 0,\; \pm 2,\; \pm 4,\; \ldots)$$
$$z \;=\; n\pi \;-\; 0.881\,i \qquad (n \;=\; \pm 1,\; \pm 3,\; \ldots)$$

3.3 Using complex numbers to evaluate trigonometric integrals

Evaluate $I \;=\; \int_0^1 e^{3x}\cos(2x)\,dx$.

We note that cos(2x) can be expressed as $Re[e^{i2x}]$ where Re means ‘the real part of’. Then, because e^{3x} is real, we can write,

$$I \;=\; \int_0^1 e^{3x} Re[e^{i2x}]\,dx \;=\; \int_0^1 Re[e^{(3+2i)x}]\,dx \;=\; Re\left[\int_0^1 e^{(3+2i)x}\,dx\right]$$

Performing the complex integral, we have,

$$\int_0^1 e^{(3+2i)x}\,dx \;=\; \left[\frac{e^{(3+2i)x}}{(3+2i)}\right]_0^1 \;=\; \frac{e^{(3+2i)}-1}{(3+2i)} \;=\; \frac{e^3[\cos(2)+i\sin(2)]-1}{(3+2i)}$$

Multiplying top and bottom by $(3-2i)$,

$$\int_0^1 e^{(3+2i)x}\,dx \;=\; \frac{[3e^3\cos(2)+2e^3\sin(2)-3] \;+\; i[3e^3\sin(2)-2e^3\cos(2)+2]}{13}$$

The required integral is now obtained by taking the real part,

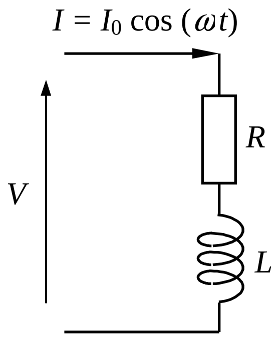
$$I \;=\; \int_0^1 e^{3x}\cos(2x)\,dx \;=\; \frac{3e^3\cos(2)+2e^3\sin(2)-3}{13}$$

Although not required, we could also take the imaginary part and get, as a bonus,

$$\int_0^1 e^{3x}\sin(2x)\,dx \;=\; \frac{3e^3\sin(2)-2e^3\cos(2)+2}{13}$$

3.4 The use of complex numbers to describe linear electric circuits

A very important application of complex numbers is in linear electric circuit theory. As an example, consider a circuit consisting of a resistor and an inductor in series :



Suppose the circuit carries an alternating current $I = I_0 \cos(\omega t)$. This means that the current varies sinusoidally with frequency $\omega/2\pi$ and has amplitude I_0. From electric circuit theory, the voltage V is given by,

$$V = IR + L\frac{dI}{dt} = I_0(R\cos\omega t - \omega L\sin\omega t)$$

Now, the sum of two sinusoids of angular frequency ω is another sinusoid, also of angular frequency ω. Hence, the expression for V can be written,

$$V = I_0 A\cos(\omega t + \phi)$$

where I_0A is the voltage amplitude and ϕ is the phase angle with respect to the current I. Using the relevant trigonometric identity, this can be written,

$$V = I_0 A(\cos\omega t\cos\phi - \sin\omega t\sin\phi)$$

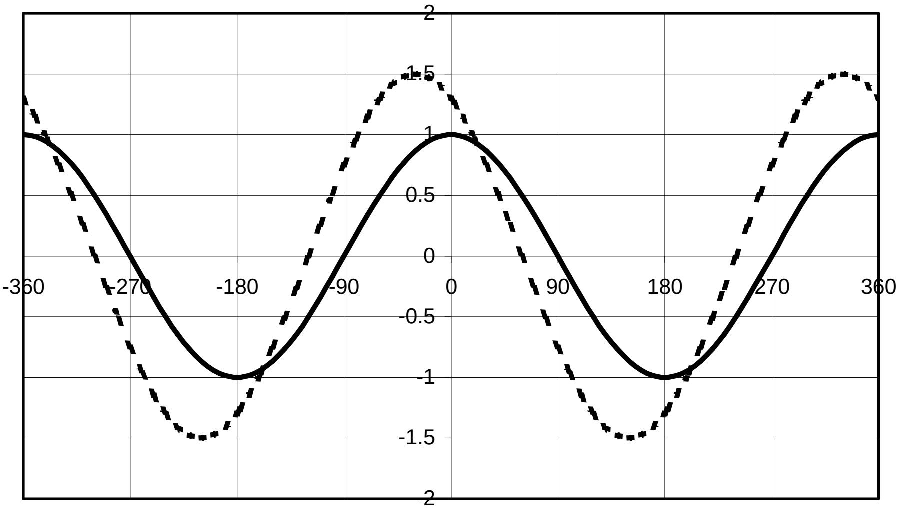
Equating the coefficients of $\cos\omega t$ and $\sin\omega t$,

$$A\cos\phi = R \ , \qquad A\sin\phi = \omega L$$

We therefore obtain the following expressions for A and ϕ,

$$A = \sqrt{R^2 + \omega^2 L^2} \ , \qquad \phi = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

The graph below shows the solution when $I_0 = 1$ amp and R, L and ω are chosen such that $A = 1.5$ and $\phi = \pi/6$ radians = 30°. Note that ωt in degrees is plotted on the time axis rather than t itself. Clearly, V (dotted line) leads I (solid line) by a phase angle of 30°.



It turns out that this type of problem can be tackled much more efficiently using complex numbers. To do this, we note that the current can be expressed as,

$$I = I_0 \cos\omega t = Re(I_0 e^{i\omega t})$$

The voltage V is therefore given by,

$$V = IR + L\frac{dI}{dt} = Re[I_0(Re^{i\omega t} + i\omega L e^{i\omega t})] = Re[I_0(R + i\omega L)e^{i\omega t}]$$

In polar form, $(R + i\omega L) = Ae^{i\phi}$ where,

$$A = \sqrt{R^2 + \omega^2 L^2} \;, \qquad \phi = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

Hence, just as before, the voltage is given by,

$$V = Re[I_0 A\, e^{i(\omega t + \phi)}] = I_0 A \cos(\omega t + \phi)$$

Now, suppose we take it as read that in any linear electric circuit the angular frequency of the voltage and current is the same, namely ω. We therefore write the voltage as,

$$V = Re[V_0 e^{i\omega t}]$$

where,

$$V_0 = I_0 A e^{i\phi} = I_0(R + i\omega L)$$

V_0 is a complex number containing everything we need to know about the voltage. It tells us that the amplitude of the voltage (in volts) is a factor $\sqrt{R^2 + \omega^2 L^2}$ times that of the current (in amps) and that the voltage leads the current by a phase angle $\phi = \tan^{-1}(\omega L/R)$.

If we define a quantity $Z = R + i\omega L$ (called the complex impedance of the circuit), we can write the relationship between voltage and current as,

$$V_0 = I_0 Z$$

which is an obvious extension of Ohm's law to sinusoidally oscillating linear electric circuits. Similar analysis can be carried out for capacitors and this opens the way to a very neat and efficient method of analysing linear electric circuits, driven by sinusoidally oscillating voltages or currents, and made up of any combination of resistors, inductors and capacitors.

3.5 Does the square root of minus one exist?

Ask this of a mathematician and he or she will mutter something like, "depends on what you mean by exist", or "does minus one exist?". More helpfully, mathematicians sometimes refine the definition such that the complex number z is defined to be an ordered pair (x, y) of real numbers x and y, written $z = (x, y)$. They then define the operation of multiplying the complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ by the relationship,

$$z_1 z_2 = (x_1, y_1) \times (x_2, y_2) = (x_1 x_2 - y_1 y_2 , x_1 y_2 + x_2 y_1)$$

With this definition complex numbers obey the laws of algebra; that is they are commutative $[z_1 z_2 = z_2 z_1]$, associative $[z_1(z_2 z_3) = (z_1 z_2) z_3]$, and distributive $[z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3]$. Complex numbers of the form $(x, 0)$ behave just like the real numbers x. The complex number $(0, 1)$ is given the symbol i and using the multiplication rule, $i^2 = (-1, 0)$. So the square root of $(-1, 0)$ in this system certainly exists and is $(0, 1)$.